Suggestions for minor radius definitions in stellarators

B.Ph. van milligen, A.L. Fraguas, J. Geiger

1 The problem

Each stellarator uses its own definition of a minor radius. For practical purposes such as representing the stellarator's experimental profiles, this is fine. However, when comparing data from different stellarators (the ISS confinement database!), this is not an unimportant issue, since the various definitions of minor radius can vary by as much as 20%, due to the strong deviations of the stellarator flux surfaces from a perfect torus.

Below, we summarize some of the methods to define the minor radius, and finish by giving a recommendation for the confinement database.

2 Definition of the minor radius of a (perfect) torus

The volume (V) and surface (S) of any flux surface (including the LCFS) can be evaluated with high precision, so it seems reasonable to attempt a definition of the minor radius based on these concepts.

A perfect torus has a circular axis and circular poloidal cross sections. The surface and volume of a perfect torus are given by:

$$\begin{aligned} S &= 4\pi^2 Rr \\ V &= 2\pi^2 Rr^2 \end{aligned} \quad (1)$$

One can either use these relations directly to define r (by choosing a value of R), or divide them to obtain:

$$r = \frac{2V}{S} \quad (2)$$

This latter definition has the advantage that it is not necessary to choose R (which may not be straightforward in a machine with a helical axis).

The problem, however, is that the magnetic surfaces are **not** perfect tori, so the question is: how can this analysis be improved to define an (effective) radius for a non-perfect torus?

3 Definition of the minor radius of a more general topological torus

If volume and surface are parametrized by a single parameter s, then

$$\int_0^s S(x)dx = V(s) \tag{3}$$

The parameter s has the dimension of a radius. Therefore, one can rename s to r without loss of generality. It follows that

$$dr = dV/S \tag{4}$$

Integrating:

$$r = \int_0^V \frac{1}{S(V')} dV' \tag{5}$$

This expression requires knowledge of the relation $S(V)$ over the whole volume in order to determine r. Another approach is obtained by rewriting Eq. (4):

$$S(r) = \frac{dV}{dr} = \frac{dV}{dS}\frac{dS}{dr} \tag{6}$$

so that:

$$\frac{S}{dS/dr} = \frac{dV}{dS} \tag{7}$$

Which is a local expression (not integral). However, in order to find the radius r one needs to make an additional assumption, namely, the dependence of S on r (or, equivalently, of V on r), as we will see below.

If one assumes that S is linear in r, or $S = cr$ (which seems reasonable for a toroidal surface), one immediately obtains a definition for the radius:

$$\frac{S}{dS/dr} = r = \frac{dV}{dS} \tag{8}$$

It is not necessary to assume anything about the volume dependence, since this is defined by Eq. 3. In contrast to Eq. (5), this definition is *local*.

However, it should be realized that this is not the only definition possible. If one makes the more general assumption that $S = cr^\alpha$, then it follows:

$$\begin{aligned} V &= \frac{c}{\alpha+1} r^{\alpha+1} \\ r &= \alpha \frac{dV}{dS} \end{aligned} \tag{9}$$

Obviously, this does not exhaust the possibilities. Generally, one could set $S = f(r)$, which should be a *monotonically increasing* function of r to make sense geometrically; however, in general, the simple linear relation between r and dV/dS is then lost.

3.1 Side note: other parametric volumes

Eq. 9 is in fact valid for any parametric surface (of a single parameter). For example, for a ball, one would simply have to assume that S depends quadratically on r: $S = cr^2$, so $\alpha = 2$ and:

$$r = 2\frac{dV}{dS} \tag{10}$$

This is consistent with the known expressions $S = 4\pi r^2$ and $V = \frac{4}{3}\pi r^3$ for a ball.

4 Definitions based on poloidal cross sections

Assume we have a flux-based coordinate system (ψ, ϕ, θ), where ψ is the flux, ϕ the toroidal cylindrical angle and θ an appropriate poloidal angle. At a given value of ϕ, the cross section is:

$$A(\psi, \phi) = \int_0^{2\pi} \int_0^{\psi} r(\psi', \theta, \phi) d\psi' d\theta = 2\pi \int_0^{\psi} \overline{r(\psi', \phi)} d\psi' \tag{11}$$

where the definition of $\overline{r(\psi, \phi)}$ (the mean radius of surface ψ at angle ϕ) is still open. We would be interested to determine the mean of this quantity over ϕ.

Note that the circumference of the poloidal cross-section A is given by

$$L(\psi,\phi) = \frac{dA}{d\psi} = 2\pi\overline{r(\psi,\phi)} \tag{12}$$

These relations inspire two numerical definitions:

- Based on the circumference: compute the circumference $L(\psi,\phi)$ for many ϕ's in order to compute the mean $\overline{L(\psi)} = 2\pi r_{\text{eff}}$.
- Based on the cross section surface area: compute the area $A(\psi,\phi)$ for many ϕ's in order to compute the mean $\overline{A(\psi)} = \pi r_{\text{eff}}^2$.

We note that these definitions are based on the idea that the cross section defined by $\phi = cst$ has some physical meaning. While this is true for tokamaks (due to symmetry), it may not be very relevant to stellarators where such symmetry is absent. In particular, these definitions may not be very appropriate for Heliacs. The reason for this lack of physical significance of poloidal cross sections is that the flux tube describing the plasma surface does not intersect the plane $\phi = cst$ perpendicularly. Expressed mathematically:

$$\vec{\nabla}\psi \cdot \vec{\nabla}\phi \neq 0 \tag{13}$$

The numerical size of this inner product determines "how good" definitions based on poloidal cross sections are: they are fine for tokamaks (where the r.h.s. is exactly zero), but bad for Heliacs, and meaningless for extreme stellarators such as Spitzer's "figure-eight" machine.

5 Consequences for scaling laws

One of the main reasons for discussing these issues is the comparison between very different machines needed to make a scaling law such as the ISS scaling of the energy confinement time. Currently, the ISS scaling law reads:

$$\tau_E = ...R^{0.64}a^{2.33}... \tag{14}$$

apart from other factors. When one replaces R and a by S and V, it it will read something like:

$$\tau_E = ...V^{1.69}S^{-1.05}... \tag{15}$$

(I did this by substituting $R \to S^2/V$ and $a \to V/S$; of course, this is cheating because one would really have to recalculate the regression with proper values of S and V to do this). This makes a lot of sense: the confinement time increases (strongly) with volume (energy content!) and decreases (linearly!) with surface (energy flux out of the plasma!).

It would be very easy to get used to scaling laws based on S and V instead of R and a, because that makes good sense, conceptually.

6 Conclusions

The definition of an effective radius is highly ambiguous for stellarators in general, and for heliacs in particular. From the above analysis, we would be inclined to give preference to a definition such as $r = dV/dS$, but we note that even this definition is not free from ambiguities. The only unambiguous definition is perhaps Eq. (5), but it is slightly awkward to manage since it requires knowledge of the full equilibrium (assuming nested flux surfaces).

We would like to suggest that the stellarator confinement data base should not only contain an effective radius r (while providing documentation on the assumptions underlying the definition of r), but also (at least) the values S (surface) and V (volume) of the (last) flux surfaces, whose definition is unambiguous. Note that one may also want to store information about the profiles of $S(\psi)$ and $V(\psi)$, or at least dV/dS at the LCFS.

An important recommendation would be to **drop** the custom, inherited from tokamaks, to make scaling laws based on an effective radius, since that could easily introduce definition errors as large as 20% or more between different machines. We note that the 'size' of a toroidal object is given by (at least) *two* parameters, due to the fact that a toroidal object is doubly connected: e.g., $\{R, r\}$, or, alternatively, $\{S, V\}$. In view of the above considerations, we believe that *multi-machine* scalings should be made on the basis of S and V, and not on the basis of the more ambiguous r and R. Possibly, such scalings might work better than the direct scalings with r and R, due to the absence of ambiguities in the definitions of S and V. It may also make physical sense, since the energy contained in the plasma should be proportional to V, while the energy loss rate should be proportional to S (in the absence of anomalous transport).